

TORIC RINGS OF NONSIMPLE POLYOMINOES

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ABSTRACT. It is known that toric ring of a simple polyomino is ring homomorphic to a edge ring of a weakly chordal bipartite graph. In this paper we identify the attached toric rings of nonsimple polyominoes which are of the form “rectangle minus rectangle”.

INTRODUCTION

Polyominoes are two dimensional objects which are originally rooted in recreational mathematics and combinatorics. They have been widely discussed in connection with tiling problems of the plane. Typically, a polyomino is plane figure obtained by joining squares of equal sizes, which are known as cells. In connection with commutative algebra, polyominoes are first discussed in [5] by assigning each polyomino the ideal of inner 2-minors or the *polyomino ideal*. The study of ideal of t -minors of an $m \times n$ matrix is a classical subject in commutative algebra. The class of polyomino ideal widely generalizes the class of ideals of 2-minors of $m \times n$ matrix as well as the ideal of inner 2-minors attached to a two sided ladder.

Let \mathcal{P} be a polyomino and K be a field. We denote by $I_{\mathcal{P}}$, the polyomino ideal attached to \mathcal{P} , in a suitable polynomial ring over K . The residue class ring defined by $I_{\mathcal{P}}$ denoted by $K[\mathcal{P}]$. It is natural to investigate the algebraic properties of $I_{\mathcal{P}}$ depending on shape of \mathcal{P} . The classes of polyominoes whose polyomino ideal is prime has been discussed in many papers. In [5] it is shown that polyomino ideal of convex polyominoes are prime. In [3] they introduced the class of *balanced* polyominoes and proved that polyomino ideal of balanced polyominoes are prime. Later in [4] it is shown that simple polyominoes are balanced. In [6], independently from [4], it is proved that polyomino ideal of simple polyominoes are prime by identifying them with toric rings of edge rings of graphs. Very recently, in [2] it is shown that polyomino ideal of polyomino of the shape “rectangle minus convex” is prime by using localization argument on the attached rings of those polyominoes.

In this paper, we identify the toric rings of polyomino ideals for the special class of nonsimple polyominoes, “rectangle minus rectangle”.

1. TORIC RINGS OF NONSIMPLE POLYOMINOES

First we recall some definitions and notation from [5]. Given $a = (i, j)$ and $b = (k, l)$ in \mathbb{N}^2 we write $a \leq b$ if $i \leq k$ and $j \leq l$. The set $[a, b] = \{c \in \mathbb{N}^2 : a \leq c \leq b\}$ is called an *interval*. If $i < k$ and $j < l$, then the elements a and b are called *diagonal*

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corners and (i, l) and (k, j) are called *anti-diagonal* corners of $[a, b]$. An interval of the form $C = [a, a + (1, 1)]$ is called a *cell* (with left lower corner a). The elements (corners) $a, a + (0, 1), a + (1, 0), a + (1, 1)$ of $[a, a + (1, 1)]$ are called the *vertices* of C . The sets $\{a, a + (1, 0)\}$, $\{a, a + (0, 1)\}$, $\{a + (1, 0), a + (1, 1)\}$ and $\{a + (0, 1), a + (1, 1)\}$ are called the *edges* of C . We denote the set of edge of C by $E(C)$.

Let \mathcal{P} be a finite collection of cells of \mathbb{N}^2 . The vertex set of \mathcal{P} , denoted by $V(\mathcal{P})$ is given by $V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C)$. For each vertex v in $V(\mathcal{P})$, we write $v = (v_1, v_2)$ where v_1 is the first and v_2 is the second coordinate of v . The edge set of \mathcal{P} , denoted by $E(\mathcal{P})$ is given by $E(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} E(C)$. Let C and D be two cells of \mathcal{P} . Then C and D are said to be *connected*, if there is a sequence of cells $\mathcal{C} : C = C_1, \dots, C_m = D$ of \mathcal{P} such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \dots, m-1$. If in addition, $C_i \neq C_j$ for all $i \neq j$, then \mathcal{C} is called a *path* (connecting C and D). The collection of cells \mathcal{P} is called a *polyomino* if any two cells of \mathcal{P} are connected, see Figure 1.

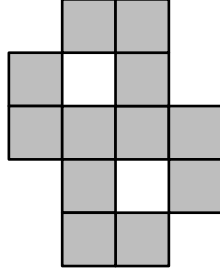


FIGURE 1. polyomino

Let \mathcal{P} be a polyomino, and let K be a field. We denote by S the polynomial ring over K with variables x_v with $v \in V(\mathcal{P})$. We write for $v = (i, j)$, we may write $x_v = x_{ij}$ if needed. Following [5] a 2-minor $x_{ij}x_{kl} - x_{il}x_{kj} \in S$ is called an *inner minor* of \mathcal{P} if all the cells $[(r, s), (r+1, s+1)]$ with $i \leq r \leq k-1$ and $j \leq s \leq l-1$ belong to \mathcal{P} . In that case the interval $[(i, j), (k, l)]$ is called an *inner interval* of \mathcal{P} . The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of \mathcal{P} is called the *polyomino ideal* of \mathcal{P} . We also set $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

For each interval $[a, b]$ we regard the polyomino $\mathcal{P}_{(a,b)}$ in the obvious way. We give Theorem 1.2 that identifies a toric ring for some specified cases of nonsimple polyominoes which is obtained by subtracting a rectangle from a bigger rectangle.

Hereafter let \mathcal{P} be a polyomino with $\mathcal{P} = \mathcal{P}_{(a,b)} \setminus \mathcal{P}_{(a'b')}$ with $(a < a' < b' < b)$. We define a map $\mu : V(\mathcal{P}) \rightarrow \mathbb{Z}$ as follows. For $v = (v_1, v_2)$ in \mathcal{P} ,

$$\mu(v) = \begin{cases} 1 & (\text{for } a_1 \leq v_1 \leq a'_1, a_2 \leq v_2 \leq a'_2) \\ 1 & (\text{for } b'_1 \leq v_1 \leq b_1, b'_2 \leq v_2 \leq b_2) \\ 2 & (\text{for } a_1 \leq v_1 \leq a'_1, b'_2 \leq v_2 \leq b_2) \\ 2 & (\text{for } b'_1 \leq v_1 \leq b_1, a_2 \leq v_2 \leq a'_2) \\ v_1 - a'_1 + 2 & (\text{for } a'_1 < v_1 < b'_1, a_2 \leq v_2 \leq b'_2) \\ v_2 - a'_2 + (b'_1 - a'_1) + 1 & (\text{for } b'_1 \leq v_1 \leq b_1, a'_2 < v_2 < b'_2) \\ b'_1 - v_1 + (b'_1 - a'_1) + (b'_2 - a'_2) & (\text{for } a'_1 < v_1 < b'_1, b'_2 \leq v_2 \leq b_2) \\ b'_2 - v_2 + 2(b'_1 - a'_1) + (b'_2 - a'_2) - 1 & (\text{for } a_1 \leq v_1 \leq a'_1, a'_2 < v_2 < b'_2). \end{cases}$$

We call $\mu(v)$ the *labelling* of v .

Example 1.1. for $a = (1, 1)$ $b = (7, 5)$, $a' = (2, 2)$ and $b' = (5, 4)$, the labelling of vertices are given as follows.

2	2	7	6	1	1	1
2	2	7	6	1	1	1
8	8			5	5	5
1	1	3	4	2	2	2
1	1	3	4	2	2	2

FIGURE 2. labelling of \mathcal{P}

We put 1 or 2 for each corners and 3, 4, ... for the other vertices anti-clockwisely.

Let $T = K[r_{v_1} s_{v_2} t_{\mu(v)} \mid v \in V(\mathcal{P})]$ be the subring of a polynomial ring $K[r_{a_1}, \dots, r_{b_1}, s_{a_2}, \dots, s_{b_2}, t_1, \dots, t_M]$ where $M = \max\{\mu(v) \mid v \in V(\mathcal{P})\}$. We define the surjective ring homomorphism $\varphi : K[\mathcal{P}] \rightarrow T$ by setting $\varphi(x_v) = r_{v_1} s_{v_2} t_{\mu(v)}$. For the sake of convenience, we write $\varphi(v) = r_{v_1} s_{v_2} t_{\mu(v)} = r_v s_v t_v$ in case it does not make readers confused. We call the *toric ideal* $J_{\mathcal{P}} = \ker \varphi$.

Theorem 1.2. Let $\mathcal{P} = \mathcal{P}_{(a,b)} \setminus \mathcal{P}_{(a'b')}$ with $(a < a' < b' < b)$. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.

Proof. First it is easy to see that $I_{\mathcal{P}} \subset \ker \varphi$. We show that every quadratic binomials belong to $J_{\mathcal{P}}$ is a generator of $I_{\mathcal{P}}$, namely an inner 2-minor in \mathcal{P} and every binomials of higher degree belongs to $I_{\mathcal{P}}$.

First, suppose that $x_{v_1} x_{v_2} - x_{v_3} x_{v_4}$ is a quadratic binomial belongs to $J_{\mathcal{P}}$ with $x_{v_1} x_{v_2} \neq x_{v_3} x_{v_4}$. Then either $(r_{v_1}, r_{v_2}, s_{v_1} s_{v_2}) = (r_{v_3}, r_{v_4}, s_{v_4} s_{v_3})$ or $(r_{v_1}, r_{v_2}, s_{v_1} s_{v_2}) = (r_{v_4}, r_{v_3}, s_{v_3} s_{v_4})$ holds. Thus, the quadratic binomial $x_{v_1} x_{v_2} - x_{v_3} x_{v_4}$ is associated with an interval. Now we show that the interval is an inner interval in \mathcal{P} . Assume

that the interval spanned by $x_{v_1}x_{v_2} - x_{v_3}x_{v_4}$ has v_1 and v_2 as diagonal corners. Also, we may assume that $r_{v_1} = r_{v_3}$ and $s_{v_1} = s_{v_4}$ hold.

Suppose $[v_1, v_2]$ is not an inner interval in \mathcal{P} . Then $\mathcal{P}_{[a', b']} \subset \mathcal{P}_{[v_1, v_2]}$ can not be happened because otherwise, from our definition of μ , one has $t_1^2 = t_{v_1}t_{v_2} \neq t_{v_3}t_{v_4} = t_2^2$. Therefore we may assume that one of $[v_1, v_3]$, $[v_1, v_4]$, $[v_3, v_2]$ or $[v_4, v_2]$ is not an inner interval in \mathcal{P} . Then from our definition of labelling we can not have $t_{v_1}t_{v_2} = t_{v_3}t_{v_4}$. Hence every degree 2 binomial in $J_{\mathcal{P}}$ is a generator of $I_{\mathcal{P}}$.

Now we show that each binomial of higher degree belongs to $I_{\mathcal{P}}$. Let $f = f^{(+)} - f^{(-)} \in \ker \varphi$ be a homogeneous binomial. Let $V_+ = v_1 \dots v_s$ be the set of vertices such that x_{v_i} appear in $f^{(+)}$ and $V_- = u_1 \dots u_t$ the vertices appear in $f^{(-)}$.

We may assume that $V_+ \cap V_- = \emptyset$, otherwise, say if $v_1 = u_1$ holds, we have $f = x_{v_1}(f^{(+)} / x_{v_1} - f^{(-)} / x_{u_1})$ and we can check $f' = f^{(+)} / x_{v_1} - f^{(-)} / x_{u_1}$ instead of f to complete the proof.

Now we show that if we have $v \in V_+ \cup V_-$ such that $\mu(v) \notin \{1, 2\}$, then we are done in this case. Assume that we have a vertex u_1 in V_- such that $u_1 \leq a'_1$ and $\mu(v) \neq 1, 2$. Since f belongs to the kernel, we have 2 vertices, say v_1 and v_2 in V_+ such that $s_{v_1}t_{v_1} | \varphi(x_{u_1})$ and $r_{v_1} | \varphi(x_{u_1})$. Let c be the rest corner of the interval spanned by v_1, v_2 and u_1 . Our rule of labelling show that the interval is an inner interval in \mathcal{P} See Figure 3 to see the situation. We obtain

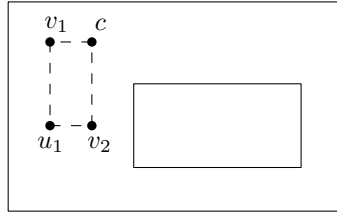


FIGURE 3.

$$\begin{aligned}
f &= f^{(+)} - f^{(-)} \\
&= x_{v_1}x_{v_2} \frac{f^{(+)}}{x_{v_1}x_{v_2}} - x_c \frac{f^{(-)}}{x_c} \\
&= (x_{v_1}x_{v_2} - x_{u_1}x_c) \frac{f^{(+)}}{x_{v_1}x_{v_2}} + x_{u_1}x_c \frac{f^{(+)}}{x_{v_1}x_{v_2}} - x_c \frac{f^{(-)}}{x_c} \\
&= (x_{v_1}x_{v_2} - x_{u_1}x_c) \frac{f^{(+)}}{x_{v_1}x_{v_2}} + x_c \left(x_{u_1} \frac{f^{(+)}}{x_{v_1}x_{v_2}} - \frac{f^{(-)}}{x_c} \right)
\end{aligned}$$

Since $x_{v_1}x_{v_2} - x_{u_1}x_c$ is an inner minor of \mathcal{P} , we have $x_{u_1}(f^{(+)} / x_{v_1}x_{v_2}) - (f^{(-)} / x_c) \in J_{\mathcal{P}}$, we can apply induction on the degree of f to complete the proof.

Now we assume every vertex in $V_{(+)}$ and $V_{(-)}$ are labelled 1 or 2. For $u_1 \in V_{(-)}$, we can find a vertex v_1 and v_2 in V_+ where $r_{u_1} = r_{v_1}$ and $s_{u_1} = s_{v_2}$. If $t_{u_1} = t_{v_1}$ or $t_{u_1} = t_{v_2}$, then using the same formula as above, we can apply induction. Now we assume that $t_{u_1} = t_1$ and $t_{v_1} = t_{v_2} = t_2$. By using similar argument, one has $u_2 \in V_-$ such that $s_{u_2} = s_{v_1}$. Since $\varphi(f^{(+)}) = \varphi(f^{(-)})$ holds, we have a vertex $u_2 \in V_-$ such

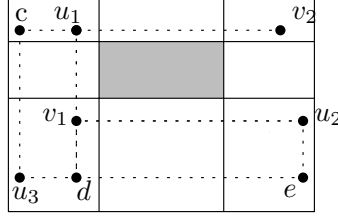


FIGURE 4.

that $t_{u_2} = t_1$. See Figure 4 for this situation. Then the interval spanned by u_1 and u_3 is an inner minor of \mathcal{P} . Let c, d be the other corners of this interval. Then,

$$\begin{aligned}
 f &= f^{(+)} - f^{(-)} \\
 &= f^{(+)} - x_{u_1}x_{u_3} \frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
 &= f^{(+)} - x_cx_d \frac{f^{(-)}}{x_{u_1}x_{u_3}} - (x_{u_1}x_{u_3} - x_cx_d) \frac{f^{(-)}}{x_{u_1}x_{u_3}}
 \end{aligned}$$

Notice that one of $t_c = t_1$ or $t_d = t_1$ holds. Assume $t_d = t_1$. Then the interval spanned by u_2 and d has v_1 as a corner vertex. Suppose that e is the rest corner vertex of this interval. Then, we have

$$\begin{aligned}
 f &= f^{(+)} - x_cx_d \frac{f^{(-)}}{x_{u_1}x_{u_3}} - (x_{u_1}x_{u_3} - x_cx_d) \frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
 &= x_{v_1} \frac{f^{(+)}}{x_{v_1}} - x_cx_dx_{u_2} \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} - (x_{u_1}x_{u_3} - x_cx_d) \frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
 &= x_{v_1} \frac{f^{(+)}}{x_{v_1}} - x_{v_1}x_cx_e \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} - (x_dx_{u_2} - x_{v_1}x_e) \left(x_c \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} \right) \\
 &\quad - (x_{u_1}x_{u_3} - x_cx_d) \frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
 &= x_{v_1} \left(\frac{f^{(+)}}{x_{v_1}} - x_cx_e \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} \right) - (x_dx_{u_2} - x_{v_1}x_e) \left(x_c \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} \right) \\
 &\quad - (x_{u_1}x_{u_3} - x_cx_d) \frac{f^{(-)}}{x_{u_1}x_{u_3}}.
 \end{aligned}$$

Since $(x_dx_{u_2} - x_{v_1}x_e)$ and $(x_{u_1}x_{u_3} - x_cx_d)$ are inner minors of \mathcal{P} and since degree of $f^{(+)} / x_{v_1} - x_cx_e f^{(-)} / x_{u_1}x_{u_2}x_{u_3}$ is less than degree of f , we can apply induction on degree of f to complete the proof. \square

Example 1.3. A toric ring of the polyomino given in Example 1.1 is identified by Theorem 1.2 as follows.

$$S/I_{\mathcal{P}} \cong K \begin{bmatrix} r_1 s_5 t_2 & r_2 s_5 t_2 & r_3 s_5 t_7 & r_4 s_5 t_6 & r_5 s_5 t_1 & r_6 s_5 t_1 & r_7 s_5 t_1 \\ r_1 s_4 t_2 & r_2 s_4 t_2 & r_3 s_4 t_7 & r_4 s_4 t_6 & r_5 s_4 t_1 & r_6 s_4 t_1 & r_7 s_4 t_1 \\ r_1 s_3 t_8 & r_2 s_3 t_8 & & & r_5 s_3 t_5 & r_6 s_3 t_5 & r_7 s_3 t_5 \\ r_1 s_2 t_1 & r_2 s_2 t_1 & r_3 s_2 t_3 & r_4 s_2 t_4 & r_5 s_2 t_2 & r_6 s_2 t_2 & r_7 s_2 t_2 \\ r_1 s_1 t_1 & r_2 s_1 t_1 & r_3 s_1 t_3 & r_4 s_1 t_4 & r_5 s_1 t_2 & r_6 s_1 t_2 & r_7 s_1 t_2 \end{bmatrix}.$$

In this paper we identify toric rings of polyominoes which are of the form “rectangle minus rectangle”. In general it is interesting but not so easy to identify toric ring for a given binomial prime ideal. The wider class of polyominoes which can be discussed about its toric ring is “rectangle minus convex”, because it is proved in [2] that their polyomino ideals are prime.

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